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SPHERICAL MEANS AND GEODESIC CHAINS ON A RIEMANNIAN MANIFOLD[1]

BY
TOSHIKAZU SUNADA

ABSTRACT. Some spectral properties of spherical mean operators defined on a Riemannian manifold are given. As an application we deduce a statistic property of geodesic chains which is interesting from the view point of geometric probability.

1. Introduction. Let M be a complete Riemannian manifold, and r a positive real number. By SM, we shall denote the tangent unit sphere bundle of M. The *spherical mean* with radius r is an operator acting on continuous functions on M by

$$(L_r f)(x) = \int_{S_x M} f(\exp rv)\, dS_x(v) \qquad (f \in C^0(M)),$$

where dS_x stands for the normalized canonical density on the unit sphere $S_x M$. When $M = \mathbf{R}^n$, this coincides with the classical spherical mean

$$(\widehat{L_r f})(x) = \int_{S^{n-1}} f(x + rv)\, dS^{n-1}(v),$$

which, as is well known, has a continuous selfadjoint extension to $L^2(\mathbf{R}^n)$. Further taking the Fourier transform of both sides, one gets an equality

$$(*) \qquad \begin{aligned} \widehat{L_r f}(\xi) &= \int_{S^{n-1}} \exp(\sqrt{-1}\, r\langle v, \xi\rangle)\, dS^{n-1}(v) \hat{f}(\xi) \\ &= \Gamma(n/2)(\|\xi\| r/2)^{-n/2+1} J_{n/2-1}(\|\xi\| r)\hat{f}(\xi), \end{aligned}$$

$J_\nu(x)$ being the Bessel function. Using here the estimate $J_\nu(x) = O(x^{-1/2})$ $(x\uparrow\infty)$, one gets an inequality

$$\int_{\mathbf{R}^n} (1 + \|\xi\|^2)^s |\widehat{L_r f}(\xi)|^2\, d\xi \leqslant c_r \int_{\mathbf{R}^n} (1 + \|\xi\|^2)^{s-(n-1)/2} |\hat{f}(\xi)|^2\, d\xi,$$

from which it follows that L_r has a continuous extension to the Sobolev spaces: $H_{(s)}(\mathbf{R}^n) \to H_{(s+(n-1)/2)}(\mathbf{R}^n)$.

These properties can be generalized to spherical means defined on Riemannian manifolds as follows.

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THEOREM A. (i) *L_r has a continuous selfadjoint extension*: $L^2(M) \to L^2(M)$ *with* $\|L_r\|_2 \leqslant 1$.

(ii) *If for any x in M, the restriction of the exponential mapping* $\exp_x$: $rS_xM \to M$ *is an immersion, then L_r has a continuous extension*: $H^{\mathrm{loc}}_{(s)}(M) \to H^{\mathrm{loc}}_{(s+(n-1)/2)}(M)$. *In particular, if, in addition, M is compact and* $\dim M \geqslant 2$, *then the operator* L_r: $L^2(M) \to L^2(M)$ *is compact, and any eigenfunction of L_r belonging to nonzero eigenvalue is smooth.*

In case r is small enough, (ii) is due to Tsujishita **[24]**, who proved in fact that the spherical mean with small radius is a *Fourier integral operator* of negative order. We will see in §2 that his idea still works well in the above situation, although L_r may be no longer a Fourier integral operator in the usual sense.

Let it be assumed that M is compact. Selfadjointness of L_r implies especially the equality $\int_M L_r f\, dg = \int_M f\, dg$ for any $f \in L^2(M)$, from which, using the general theory of Markov processes, one can equip a probability measure on the infinite product $M \times M \times M \times \ldots$ such that the shift transformation is measure preserving and the coordinate process is a Markov process whose transition operator is just L_r. Ergodicity of this dynamical system is equivalent to the statement that 1 is simple eigenvalue of L_r. In this view, L_r will be called *ergodic* if the equality $L_r f = f$ ($f \in L^2(M)$) implies that f is constant.

Ergodicity of L_r means something more. To explain this, let us give here a notion of geodesic chains. Let k be a positive integer or ∞. We will call a continuous mapping c of the interval $[0, k]$ into a complete Riemannian manifold M a *geodesic chain* of length k if each restriction $c_i = c|[i-1, i]$ ($i = 1, 2, \ldots$) is a geodesic curve in M. If, in addition, every c_i has a common length r, then such a curve will be called an r-geodesic chain. We denote by $\mathcal{C}_k(r; M)$ the set of all r-geodesic chains of length k in M. As will be seen in §3, the set $\mathcal{C}_k(r; M)$ is naturally identified with the fiber bundle S^kM whose fiber S^k_xM is the k-product $S_xM \times S_xM \times \cdots \times S_xM$, so that $\mathcal{C}_k(r; M)$ has a measure which comes from the fiber product measure on S^kM. Suppose again M is compact. A geodesic chain $c \in \mathcal{C}_\infty(r; M)$ will be called *uniformly distributed* if for any Jordan measurable subset J in M, the time average

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \chi_J(c(k)),$$

χ_J being the defining function, exists and equals the space average $\int_J dg / \int_M dg$ (cf. Weyl **[25]**)

THEOREM B. *If L_r is ergodic, then almost all r-geodesic chains are uniformly distributed. Here "almost all" means "except for a set of measure zero in $\mathcal{C}_\infty(r; M)$."*

The proof relies upon a property of the measure on $\mathcal{C}_\infty(r; M)$, which guarantees the existence of a geometric model of Markov process on $\mathcal{C}_\infty(r; M)$ with transition operator L_r.

By Theorem B, the ergodicity of L_r implies that there exists an r-geodesic chain c such that the set $\{c(k); k = 1, 2, \ldots\}$ is dense in M. If M is a compact homogeneous space, the converse holds (cf. [20]). In the general case, the situation seems more complicated. But as an application of Theorem A, we can prove

THEOREM C. *If* dim $M \geqslant 2$, *and for any* x *in* M, *the mapping* $\exp_x\colon rS_xM \to M$ *is an immersion, then* L_r *is ergodic.*

In the course of proof, we require the fact that under the above condition, any two points in M can be joined by an r-geodesic chain of finite length.

As particular cases, we have

COROLLARY. *If one of the following conditions is satisfied, then* L_r *is ergodic.*
(i) *The radius* r *is smaller than the injective radius of* M.
(ii) M *has no conjugate point* (r *is arbitrary*).

Furthermore, we get some information about discrete spectrum.

THEOREM D. *Under the same conditions as Theorem* C:
(i) -1 *is not an eigenvalue*;
(ii) *there are infinitely many positive eigenvalues.*
If, in addition, M *has no geodesic loop with length* r, *then there are infinitely many negative eigenvalues.*

The first assertion implies the mixing property of the dynamical system. The fundamental idea of the proof of (ii) is due to M. Gromov.

We now return to the equality (*). The Fourier inversion gives rise to a formal expansion of L_r with respect to r:

$$L_r \sim I - (r^2/2n)\Delta + (r^4/8n(n+2))\Delta^2 + \dots .$$

In §5, we observe that spherical means on Riemannian manifolds have also analogous expansions, which, as by-product, provide us with an approximation theorem for the fundamental solution for the heat equation by an iteration of L_r:

THEOREM E. *Let* M *be compact, and let* K_t *be the fundamental solution for the heat equation* $(\partial/\partial t + \Delta)u = 0$. *Then for any* $f \in L^2(M)$, *the iteration*

$$L^N_{r(t,N)}f \qquad \left(r(t,N) = \sqrt{2nt/N}\,\right),$$

converges to K_tf *in* L^2*-sense and in a.e. sense as* $N \uparrow \infty$.

The formal expansion of L_r allows us also to prove the following.

THEOREM F. *If* M *is a symmetric space with rank* $\geqslant 2$, *then* L_r *is ergodic for any* $r > 0$.

In the case of rank one, we can give a condition for radius r in order that L_r is ergodic.

We should point out that some of the results about L_r had been obtained by P. Günther [9] when M is a harmonic Riemannian space, and that an expansion theorem for a similar mean valued operator has been proved by A. Gray and T. J. Willmore [7].

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2. Spherical means. Let M be a connected n-dimensional complete Riemannian manifold, and $\pi\colon SM \to M$ its tangent unit sphere bundle. The canonical density associated with the metric of M is denoted dg. The bundle SM possesses a measure given by the density dSM which takes the form

$$\int_{SM} f\, dSM = \int_M dg(x) \int_{S_xM} f(v)\, dS_x(v).$$

The density dSM coincides with the one associated with the standard metric on SM (see [18]).

Let L_r be the spherical mean with radius r. It is easy to see that L_r is an operator of the space of bounded continuous functions into itself with $\sup|L_rf| \leqslant \sup|f|$.

LEMMA 2.1. *L_r can be extended to a continuous operator $L_r\colon L^p(M) \to L^p(M)$, $p \geqslant 1$, with $\|L_r\|_p \leqslant 1$. Further, in case $p = 2$, the extension is a selfadjoint operator of $L^2(M)$.*

PROOF. Let $\{\varphi_t\}$ $(-\infty < t < \infty)$ be the geodesic flow on SM, and $\pi_*\colon C^0(SM) \to C^0(M)$ the fiber integral

$$(\pi_* h)(x) = \int_{S_xM} h\, dS_x.$$

Then we easily find $L_r = \pi_* \varphi_r^* \pi^*$, where φ_r^*, π^* denote the pull backs by φ_r, π, respectively. Since $\{\varphi_t\}$ is a volume preserving flow (see [18]), we get

$$\begin{aligned}\int_M L_rf \cdot h\, dg &= \int_M h(x)\, dg(x) \int_{S_xM} \varphi_r^* \pi^* f\, dS_x \\ &= \int_{SM} \varphi_r^*(\pi^* f \cdot \varphi_{-r}^* \pi^* h)\, dSM \\ &= \int_{SM} \pi^* f \cdot \varphi_{-r}^* \pi^* h\, dSM = \int_M f \cdot L_r h\, dg,\end{aligned}$$

which proves the last half of our assertion. The rest follows from the estimates $\|\pi^*\|_p = 1$, $\|\pi_*\|_p \leqslant 1$.

REMARK. In his paper [9], P. Günther considered an operator relating to a mean valued operator treated by Ruse, Walker and Willmore [16], which takes the form

$$(M_rf)(x) = \int_{rS_xM} f(\exp v)\, d\sigma'_x,$$

where $d\sigma'_x$ is a volume element on the r-sphere satisfying $\exp_x^*(dg) = dr \wedge d\sigma'_x$. He proved that M_r is selfadjoint. But the mean valued operator $\tilde{L}_r$ of the form

$$(\tilde{L}_rf)(x) = M_r(1)(x)^{-1}(M_rf)(x)$$

is not in general selfadjoint. If M is a harmonic Riemannian space, $\tilde{L}_r$ coincides with L_r.

We now proceed to the proof of Theorem A(ii). By ∇, we shall denote the covariant differentiation associated with the metric. We assume hereafter that for any x in M, the mapping $\exp_x\colon rS_xM \to M$ is an immersion. We should note that this assumption is equivalent to the statement that the mapping $\pi_r\colon SM \to M \times M$ given by $\pi_r(v) = (\pi(v), \exp rv)$ is an immersion, or that for any $v \in SM$, the point $\exp rv$ is not conjugate to $\pi(v)$ along the geodesic $\exp tv$ $(0 \leqslant t \leqslant r)$.

The following is easy to see.

LEMMA 2.2. *The normal bundle $N(\pi_r)$ of the immersion π_r is given by*

$$\bigcup_{v \in SM} R(v \oplus -\varphi_r(v)).$$

To prove Theorem A, let $\{U_\alpha\}$ be a locally finite open covering of M such that the inverse image $\pi_r^{-1}(U_\alpha \times U_\beta)$ is a finite disjoint union of open sets $\{W_i\}$ and each restriction $\pi_r\colon W_i \to U_\alpha \times U_\beta$ is injective. Then using partitions of unity $\{\varphi_\alpha\}$ associated with $\{U_\alpha\}$, we have

$$L_rf = \sum_{\alpha,\beta} \varphi_\alpha L_r \varphi_\beta f \quad \text{for } f \in C_0^\infty(M).$$

We will prove that the operator $L_{\alpha\beta} = \varphi_\alpha L_r \varphi_\beta\colon C_0^\infty(U_\beta) \to C_0^\infty(U_\alpha)$ is written as a finite sum of Fourier integral operators possibly with distinct Lagrange manifolds. To this end, let $K_{\alpha\beta}$ be the distribution kernel of $L_{\alpha\beta}$, which has the form

$$\begin{aligned}\langle K_{\alpha\beta}, h\rangle &= \int_{SM} \pi_r^*(\varphi_\alpha\varphi_\beta h)\, dSM \\ &= \sum_i \int_{W_i} \pi_r^*(\varphi_\alpha\varphi_\beta h)\, dSM \qquad \big(h \in C_0^\infty(U_\alpha \times U_\beta)\big).\end{aligned}$$

Hence it suffices to show that each distribution $K^i_{\alpha\beta}$ given by

$$\langle K^i_{\alpha\beta}, h\rangle = \int_{W_i} \pi_r^*(\varphi_\alpha\varphi_\beta h)\, dSM$$

belongs to the class $I^{-(n-1)/2}(U_\alpha \times U_\beta, \Lambda_{W_i})$ (see [11] for the notation), where $\Lambda_{W_i} = T_{W_i}(U_\alpha \times U_\beta) \setminus 0$ is the conormal bundle of W_i minus zero section. For this, consider W_i as a submanifold of $U_\alpha \times U_\beta$. We may assume there exists a local coordinate $(x_1, \ldots, x_{2n})$ of $U_\alpha \times U_\beta$ such that $W_i = \{(x_1, \ldots, x_{2n});\ x_{2n} = 0\}$. Put $dSM|_{W_i} = f dx_1 \cdots dx_{2n-1}$. Then

$$\begin{aligned}\langle K^i_{\alpha\beta}, h\rangle &= \int_{W_i} \varphi_{\alpha\beta}(x_1, \ldots, x_{2n-1}, 0) h(x_1, \ldots, x_{2n-1}, 0) f dx_1 \cdots dx_{2n-1} \\ &= (2\pi\sqrt{-1}\,)^{-1} \int_{\mathbf{R}^n \times \mathbf{R}} h(x_1, \ldots, x_{2n}) a(x_1, \ldots, x_{2n}, \theta) \\ &\qquad\qquad \cdot \exp(x_{2n}\theta) dx_1 \cdots dx_{2n}\, d\theta\end{aligned}$$

where $\varphi_{\alpha\beta} = \varphi_\alpha \cdot \varphi_\beta$, $a(x, \theta) = f(x_1, \ldots, x_{2n-1})\varphi_{\alpha\beta}(x_1, \ldots, x_{2n})$. Hence, $K^i_{\alpha\beta}$ is expressed as an oscillatory integral which belongs to $I^m(U_\alpha \times U_\beta, \Lambda)$ with

$$\Lambda = \{(x_1, \ldots, x_{2n-1}, 0; \theta dx_{2n})\}, \quad m = -\tfrac{1}{2}(n-1).$$

We now let $L^i_{\alpha\beta}\colon \mathfrak{D}(U_\beta) \to \mathfrak{D}'(U_\alpha)$ be a linear mapping with the distribution kernel $K^i_{\alpha\beta}$. Then the regularity of L_r reduces to that of $L^i_{\alpha\beta}$. Noting that the mappings

$$N(\pi_r) \setminus 0 \to TM \setminus 0, \qquad N(\pi_r) \setminus 0 \to TM \setminus 0,$$
$$c(v \oplus -\varphi_r(v)) \mapsto cv, \quad c(v \oplus -\varphi_r(v)) \mapsto -c\varphi_r(v)$$

are two-fold coverings, we see that the restrictions to Λ_{W_i} of the natural projections: $T^*(U_\alpha \times U_\beta) \to T^*U_\alpha$, T^*U_β are local diffeomorphisms, which implies that $L^i_{\alpha\beta}$ is extended to: $H^{\text{comp}}_{(s)}(U_\alpha) \to H^{\text{comp}}_{(s+(n-1)/2)}(U_\beta)$ (see **[11]**). This proves Theorem A(ii).

REMARK. If there is no closed geodesic with length $2r$ then L_r is a F.I.O. in the sense of Hörmander, because $N(\pi_r) \setminus 0$ has no intersection in $TM \times TM$, and hence the conormal bundle $\Lambda(SM)$ is a Lagrange manifold in $T^*M \times T^*M$.

3. Geodesic chains. The set of all geodesic segments with length r in a complete manifold M is just $\mathcal{C}_1(r; M)$, which is identified with the total space SM by a bijection $\tau_1\colon SM \to \mathcal{C}_1(r; M)$ defined by $\tau_1(v)(t) = \exp trv$, $0 \leq t \leq 1$. Using this identification, we can introduce a measure P_1 onto the set $\mathcal{C}_1(r; M)$, which is natural from a geometric point of view. For instance, when $M = \mathbf{R}^n$, this is just the usual measure on the set of oriented line segments with constant length, introduced in the theory of geometric probability **[17]**.

For general k, we can equip a measure on $\mathcal{C}_k(r; M)$ in a similar way. To explain this, let S^kM be the fiber bundle on M whose fiber at x is the product $S_xM \times \cdots \times S_xM$ (k-times). The set $\mathcal{C}_k(r; M)$ is then identified with S^kM as follows. For a chain $c = (c_1, \ldots, c_k)$, $c_i = c|[i-1, i]$, one can associate a series of tangent vectors $(v_1, v_2', \ldots, v_k^{(k-1)})$ such that $v_i^{(i-1)}$ is the unit velocity vector of c_i at the point $c(i-1)$. Translating the vector $v_i^{(i-1)}$ parallelly along the curve $(c_1, \ldots, c_{i-1})$, one can get a sequence $(v_1, \ldots, v_k) \in S^k_xM$. It is easy to see that this correspondence is bijective. We let $\tau_k\colon S^kM \to \mathcal{C}_k(r; M)$ be the inverse of this bijection. Note that the subset $\mathcal{C}_k(r)_x$ in $\mathcal{C}_k(r)$, which consists of all r-geodesic chains c with $c(0) = x$, corresponds to the fiber S^k_xM.

Define a probability measure on $\mathcal{C}_k(r)_x$ to be the product of normalized measure on the Euclidean sphere S_x. We shall denote it by $P_{k,x}$. We can then equip a measure P_k on $\mathcal{C}_k(r)$ so that the following relation holds:

$$\int_{\mathcal{C}_k} f\, dP_k = \int_M dg(x) \int_{\mathcal{C}_{k,x}} f\, dP_{k,x}.$$

Consider a mapping $p_k\colon \mathcal{C}_\infty(r)_x \to \mathcal{C}_k(r)_x$ defined by $p_k(c) = c|[0, k]$. Since p_k corresponds, in the identification, to the projection $S^\infty_xM \to S^k_xM$, the inverse image $p_k^{-1}(c)$ is identified with the product $\Pi^\infty_{k+1} S_xM$, which has a product measure to be denoted $P_x(\ |c)$. We easily observe

$$P_{\infty,x}(\mathbf{A}) = \int_{\mathcal{C}_{k,x}} P_x\big(\mathbf{A} \cap p_k^{-1}(c)|c\big)\, dP_{k,x}(c)$$

for any measurable set $\mathbf{A}$ in $\mathcal{C}_\infty(r)_x$.

It is worthwhile to note that the compact open topology of $\mathcal{C}_k(r)$ coincides with the induced one from S^kM, and that if $\mathbf{A}$ is measurable in $\mathcal{C}_k(r)_x$ with measure 1, then $\mathbf{A}$ is dense in $\mathcal{C}_k(r)_x$.

From now on, let us assume M is compact and $\int_M dg = 1$, so that P_k is a probability measure. We write, for brevity, $\mathcal{C}_x(r)$, $\mathcal{C}(r)$, P_x, P for $\mathcal{C}_\infty(r;\ M)_x$, $\mathcal{C}_\infty(r;\ M)$, $P_{\infty,x}$, P_∞, respectively. Define a mapping π_k: $\mathcal{C}(r) \to M$ by $\pi_k(c) = c(k)$. Let $\mathfrak{B}$ express the σ-ring generated by $\pi_k^{-1}(A)$ $(k = 1, 2, \ldots)$ where A runs over all Borel subsets in M. The shift T is the mapping of $\mathcal{C}(r)$ into itself given by $(Tc)(t) = c(t+1)$. Since $\pi_{k+1} = \pi_k \circ T$, T is $\mathfrak{B}$-measurable.

LEMMA 3.1. *Let $c \in \mathcal{C}_k(r;\ M)$, and $\mathbf{A} \in \mathfrak{B}$. Then*

$$P_x\big(T^{-k}\mathbf{A} \cap p_k^{-1}(c)|c\big) = P_{c(k)}(\mathbf{A}).$$

PROOF. Suppose $c = \tau_k(v_1^0, \ldots, v_k^0)$. On the subset $(v_1^0, \ldots, v_k^0) \times \Pi_{k+1} S_xM$ in S_xM, the kth iteration of T is expressed as $(v_1^0, \ldots, v_k^0, v_{k+1}, \ldots) \mapsto (P_cv_{k+1}, P_cv_{k+2}, \ldots)$, P_c being the parallel translation along the curve c, which deduces

$$P_x\big(T^{-k}\mathbf{A} \cap p_k^{-1}(c)|c\big) = \text{measure of } \big\{(v_{k+1}, v_{k+2}, \ldots);\ (P_cv_{k+1}, \ldots) \in \mathbf{A}\big\}.$$

Since the parallel translation is volume preserving, the last expression is equal to $P_{c(k)}(\mathbf{A})$ as desired.

The shift operator and the spherical mean L_r have a close connection with each other:

LEMMA 3.2. *For $\mathbf{A} \in \mathfrak{B}$, put $P_x(\mathbf{A}) = P_x(\mathbf{A} \cap \mathcal{C}_x(r))$. Then the function $x \mapsto P_x(\mathbf{A})$ belongs to $L^1(M)$, and $P_x(T^{-1}\mathbf{A}) = (L_rP.(\mathbf{A}))(x)$.*

PROOF. The first assertion is obvious. The second follows from the above lemma. Indeed

$$\begin{aligned} P_x(T^{-1}\mathbf{A}) &= \int_{\mathcal{C}_{1,x}} P_x\big(T^{-1}\mathbf{A} \cap p_1^{-1}(c)|c\big)\, dP_{1,x}(c) \\ &= \int_{S_xM} P_{\exp(rv)}(\mathbf{A})\, dS_x(v) = (L_rP.(\mathbf{A}))(x). \end{aligned}$$

Let $\mathbf{A} \in \mathfrak{B}$. By definition of P,

$$P(\mathbf{A}) = \int_M dg(x) \int_{\mathcal{C}_x(r)} \chi_{\mathbf{A}}\, dP_x = \int_M P_x(\mathbf{A})\, dg(x).$$

Hence from Lemma 2.1, we find

$$\begin{aligned} P(T^{-1}\mathbf{A}) &= \int_M P_x(T^{-1}\mathbf{A})\, dg(x) = \int_M L_r(P.(\mathbf{A}))\, dg \\ &= \int_M P_x(\mathbf{A})\, dg(x) = P(\mathbf{A}). \end{aligned}$$

Thus we have proved

PROPOSITION 3.3. *The triple $(\mathcal{C}(r), P, T)$ is a dynamical system.*

REMARK. (i) Instead of the set $\mathcal{C}_\infty(r; M)$, one can consider a set $\mathcal{C}_{-\infty,\infty}(r; M)$ consisting of geodesic chains $c: (-\infty, \infty) \to M$. In the same way, we can introduce a measure on this set such that the shift is a bijective measure preserving transformation.

(ii) Lemma 3.2 implies, for any Borel set A, $P_x(\pi_k^{-1}(A)) = (L_r^k \chi_A)(x)$. Therefore the kth iteration L_r^k of spherical mean is an integral operator associated with the double fibering

$$\begin{array}{ccc} & S^kM & \\ \pi_0 \swarrow & & \searrow \pi_k \\ M & & M, \end{array}$$

that is, $L_r^k f = \int_{S_x^k} \pi_k^* f \, dP_{x,k}$.

As is usual, the dynamical system $(\mathcal{C}(r), P, T)$ is said to be *ergodic* if any invariant $\mathfrak{B}$-measurable set has measure zero, or its complement does. Ergodicity is equivalent to either one of the following conditions.

(i) For any Borel set A in M,

$$\lim_{N\to\infty} \frac{1}{N}\left(\sum_{k=0}^{N-1} \chi_A(c(k))\right) = \int_A dg \quad \text{a.e. } c.$$

(ii) For any $f \in L^2(M)$,

$$\lim_{N\to\infty} \frac{1}{N}\left(\sum_{k=0}^{N-1} L_r^k f(x)\right) = \int_M f\, dg \quad \text{a.e. } x.$$

(iii) If $L_r f = f$, $f \in L^2(M)$, then $f =$ constant.

For instance, the equivalence of (i) and ergodicity comes from

$$\frac{1}{N}\sum_{k=0}^{N-1} \chi_{\pi_l^{-1}(A)}(T^k c) = \frac{N+l}{N}\frac{1}{N+l}\left(\sum_{k=0}^{N+l-1} \chi_A(c(k)) - \sum_{k=0}^{l-1} \chi_A(c(k))\right).$$

Further, the equivalence of (i) and (ii) comes from the equality

$$\int_{\mathcal{C}(r)} \left(\chi_{\pi_0^{-1}(A)} \circ T^k\right)\chi_{\pi_0^{-1}(B)}\, dP = \int_M L_r^k \chi_A \cdot \chi_B\, dg,$$

which is deduced as

$$\begin{aligned} \text{the left-hand side } &= \int_{\mathcal{C}(r)} \chi_{\pi_k^{-1}(A)\cap \pi_0^{-1}(B)}\, dP \\ &= \int_M P_x\left(\pi_k^{-1}(A) \cap \pi_0^{-1}(B)\right) dg(x) \\ &= \int_M P_x\left(\pi_k^{-1}(A)\right)\cdot \chi_B(x)\, dg(x) \\ &= \text{the right-hand side.} \end{aligned}$$

We are in a position to prove Theorem B. Suppose that $(\mathcal{C}(r), P, T)$ is ergodic. We will construct a measurable set **A** in $\mathcal{C}(r)$ with $P(\mathbf{A}) = 1$ such that any geodesic chains in **A** are uniformly distributed. Choose a system of C^∞-functions $\{f_i\}$ $(i = 1, 2, \ldots)$ such that any bounded Riemannian integrable function can be approximated from above and below by finite linear combinations of $\{f_i\}$ in the sense that given f we can find, for any $\varepsilon > 0$, $c_1, \ldots, c_m, d_1, \ldots, d_m$ $(\in \mathbf{R})$ such that

$$\sum c_i f_i \leqslant f \leqslant \sum d_i f_i, \quad \int_M \sum d_i f_i - \sum c_i f_i \, dg < \varepsilon.$$

For instance we may choose for $\{f_i\}$ a complete orthonormal basis of eigenfunctions of the Laplacian. The subset

$$\mathbf{A} = \left\{ c \in C(r); \ \lim_{N \to \infty} \frac{1}{N} \left(\sum_{k=0}^{N-1} f_i(c(k)) \right) = \int_M f_i \, dg \text{ for any } i \right\}$$

satisfies our demand.

4. Ergodicity and eigenvalues of spherical means. Let M be a complete manifold. Let k be a positive integer. We endow M with an equivalence relation by setting $x \sim_k^r y$ if and only if x and y are joined by an r-geodesic chain of length kl for an integer l. The following is obvious.

LEMMA 4.1 (MAXIMAL PRINCIPLE). *Let f be a continuous function on M satisfying $L_r^k f = f$. Suppose that f attains the maximum, and that an equivalence class of the relation $\sim_k^r$ containing a point at which f takes the maximum is dense in M. Then* $f \equiv$ constant. *In particular, if M is compact, and if any two points of M are joined by an r-geodesic chain, then any continuous eigenfunction of L_r with eigenvalue* 1 *must be constant.*

To prove Theorem C, it is enough to see the following (taking account of the above and Theorem A(ii)).

LEMMA 4.2. *Suppose* $\dim M \geqslant 2$, *and the mapping π_r is immersion. Then any two points of M can be joined by an r-geodesic chain of even length.*

PROOF. We will show that any equivalence classes of the relation $\sim_2^r$ are open. We make use of the following notations:

$B_\varepsilon(x) = \{y \in M; d(x, y) < \varepsilon\}$,
$\Delta = \{(u, u) \in S_xM \times S_xM\}$,
$\Delta_\delta = \{(u, v) \in S_xM \times S_xM; g(u, v) > 1 - \delta\}$,
$\partial\Delta_\delta = \{(u, v) \in S_xM \times S_xM; g(u, v) = 1 - \delta\}$.

Let $x \in M$, and let F: $S_xM \times S_xM \to M$ be a mapping given by $F(u, v) = \pi_2(\tau_2(u, -v))$. From the assumption, we easily see that there exist small positive numbers ε, δ such that

$$F(\partial\Delta_\delta) \subset M \setminus B_\varepsilon(x), \qquad F(\Delta_\delta) \cap (B_\varepsilon(x) \setminus x) = F(\Delta_\delta \setminus \Delta) \cap (B_\varepsilon(x) \setminus x).$$

Since the set $A = F(\overline{\Delta}_\delta) \cap (B_\varepsilon(x) \setminus x)$ is closed in the connected space $B_\varepsilon(x) \setminus x$, we have only to prove that A is open in $B_\varepsilon(x) \setminus x$. In fact, this being the case, the point x and any other point in $B_\varepsilon(x)$ can be joined by an r-geodesic chain of length two, so that the open neighborhood $B_\varepsilon(x)$ of x is contained in the equivalence class of x.

Our claim comes from the assertion that the restriction F: $\Delta_\delta - \Delta \to M$ is a submersion, which is proved in the following way. Let $(u, v) \in \Delta_\delta \setminus \Delta$, and let dF: $T_uS_xM \oplus T_uS_xM \to T_{F(u,v)}M$ be the differential of F at (u, v). We denote by

$v^* \in T_{\exp(ru)}M$ the parallel translation of v along the geodesic $\tau_1(u)$, and by $v^{**} \in T_{F(u,v)}M$ that of v along the chain $\tau_1(u, -v)$. Using Gauss' lemma, we get

$$dF(0 \oplus T_v S_x M) = \text{the orthogonal complement of } v^{**} \text{ in } T_{F(u,v)}M.$$

To compute $dF(X \oplus 0)$, $(X \in T_u S_x M)$, we choose a curve $u(s)$ in $S_x M$ with $u(0) = u$, $du(0)/ds = X$, and consider a family of geodesics $\{c_s(t)\}_s$ given by $c_s(t) = \tau_2(u(s), -v)(t+1)$ $(0 \leqslant t \leqslant 1)$. The vector field $J_X(t) = (\partial/\partial s)|_{s=0}\, c_s(t)$ is then a Jacobi field along the curve $c(t) = c_0(t)$, satisfying

$$dF(X \oplus 0) = J_X(1), \quad d_u(\pi \circ \varphi_r)(X) = J_X(0).$$

Further we have

$$\begin{aligned}(\nabla_{\dot c} J(0), \dot c(0)) &= g\left(\frac{D}{\partial t}\bigg|_{t=0} \frac{\partial}{\partial s}\bigg|_{s=0} c_s(t), \dot c(0)\right)\\ &= g\left(\frac{D}{\partial s}\bigg|_{s=0} \frac{\partial}{\partial t}\bigg|_{t=0} c_s(t), \dot c(0)\right)\\ &= \frac{1}{2}\frac{\partial}{\partial s}\bigg|_{s=0} g\left(\frac{\partial}{\partial t}\bigg|_{t=0} c_s(t), \frac{\partial}{\partial t}\bigg|_{t=0} c_s(t)\right)\\ &= 0,\end{aligned}$$

because $g(\dot c_s(0), \dot c_s(0)) \equiv r^2$. We now suppose that $g(J_X(1), v^{**}) = 0$ for any X in $T_u S_x M$. Since $(d^2/dt^2) g(J_X(t), \dot c(t)) = 0$, we have constants a, b such that

$$g(J_X(t), \dot c(t)) = at + b.$$

But $a + b = g(J_X(1), \dot c(1)) = -g(J_X(1), v^{**}) = 0$, and $a = g(\nabla_{\dot c} J(0), \dot c(0)) = 0$, which implies

$$g(J_X(0), v^*) = -r^{-1} g(J_X(0), \dot c(0)) = 0.$$

On the other hand, we have from Gauss' lemma,

$$\{J_X(0); X \in T_u S_x M\} = \text{the orthogonal complement of } \varphi_r(u) \text{ in } T_{\exp(ru)}M,$$

which contradicts the fact that $\varphi_r(u)$ is not any constant multiple of v^*. Hence we can find a vector $X \in T_u S_x M$ such that $J_X(1)$ is not contained in the hyperplane $dF(0 \oplus T_v S_x M)$, which implies $\operatorname{Im} dF = T_{F(u,v)}M$, as desired.

The above lemma leads also to Theorem D(i). Indeed, if $L_r f = -f$, then $f \in C^\infty(M)$, and $L_r^2 f = f$; hence by the maximum principle, f must be constantly zero.

As a corollary, we have the following which strengthens the assertion for ergodicity.

PROPOSITION 4.3. *Under the same assumption as Theorem* C, *the dynamical system* $(\mathcal{C}(r), P, T)$ *has the mixing property, namely, for any pair* **A**, **B** *in* $\mathfrak{B}$, $\lim_{N\to\infty} P(T^{-N}\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A}) \cdot P(\mathbf{B})$.

PROOF. Our assertion is obviously equivalent to: For any $f, h \in L^2(M)$,

$$\lim_{N\to\infty} \int_M L_r^N f \cdot h \, dg = \int_M f \, dg \int_M h \, dg.$$

This comes from a stronger assertion:

$$\lim_{N\to\infty} L_r^N f = \int_M f \, dg \quad \text{in } L^2\text{-sense},$$

which, looking at the L^2-expansion of f into eigenfunctions, is easily deduced.

We now focus our attention on eigenvalues other than ± 1. We continue to assume that the mapping π_r: $SM \to M \times M$ is an immersion. Note that the singular support of the distribution kernel of L_r is just the image of π_r, therefore from smoothness of eigenfunctions, it follows immediately that L_r has infinitely many eigenvalues.

To prove the existence of infinitely many positive eigenvalues, it suffices to see that, for any positive integer N, there exist smooth functions $\Phi_1, \ldots, \Phi_N$ such that
$\langle \Phi_i, \Phi_j \rangle = 0$ for $i \neq j$,
$\langle L_r\Phi_i, \Phi_j \rangle = 0$ for $i \neq j$,
$\langle L_r\Phi_i, \Phi_i \rangle > 0$ for any i,
where $\langle\ ,\ \rangle$ denotes the L^2-inner product $\int_M(\ ,\ )\,dg$. For this we first show that we can take a set $\{x_1, \ldots, x_N, y_1, \ldots, y_N\}$ so that

$$\exp(rS_{x_i}M) \ni y_i \quad \text{for any } i, \quad \exp(rS_{x_i}M) \not\ni x_j \quad \text{for } i \neq j,$$
$$\exp(rS_{y_i}M) \not\ni y_j \quad \text{for } i \neq j, \quad \exp(rS_{x_i}M) \not\ni y_j \quad \text{for } i \neq j.$$

This is obviously possible if r is small enough. In the general case, we have to choose x_i, y_i somewhat carefully. The following is useful in the argument.

LEMMA 4.4. *Let* $x \neq y$. *Suppose* $(\exp(rS_xM)) \cup (\exp(2rS_xM)) \not\ni y$. *Then the immersed manifold* $\exp(rS_xM)$ *intersects transversally with* $\exp(rS_yM)$.

PROOF. Suppose that $(\exp(rS_xM)) \cap (\exp(rS_yM)) \neq \varnothing$, and that at a point z in the intersection, $\exp(rS_xM)$ does not transversally intersect with $\exp(rS_yM)$. This implies these tangent spaces at z coincide with each other, or equivalently, there are vectors $u \in S_xM$, $v \in S_yM$ such that $\varphi_r(u) = \pm\varphi_r(v)$. Since $x \neq y$, we have $\varphi_r(u) = -\varphi_r(v)$, which means $y = \exp(2ru)$, and this is a contradiction.

This lemma guarantees that the set $\exp(rS_xM) \setminus \exp(rS_yM)$ is open dense in $\exp(rS_xM)$.

We now take points $x_1 \neq y_1$ with $y_1 \in \exp(rS_{x_1}M)$, and take a point x_2 in the complement of the union

$$\exp(rS_{x_1}M) \cup \exp(2rS_{x_1}M) \cup \exp(rS_{y_1}M) \cup \exp(2rS_{y_1}M).$$

From what has been said, we can take a point y_2 in

$$\exp(rS_{x_2}M) \setminus \big(\exp(rS_{x_1}M) \cup \exp(rS_{y_1}M)\big).$$

Proceeding with this argument, we can get a set $\{x_i, y_i\}$ with the desired property.

In order to construct functions Φ_i, take C^∞-functions φ_i, ψ_i ($i = 1, \ldots, N$) such that $\varphi_i \geq 0$, $\psi_i \geq 0$, $\varphi_i > 0$ on $B_\varepsilon(x_i)$, $\psi_i > 0$ on $B_\varepsilon(y_i)$, $\operatorname{supp}\varphi_i = \overline{B_\varepsilon(x_i)}$, $\operatorname{supp}\psi_i = \overline{B_\varepsilon(y_i)}$. Choosing a small $\varepsilon > 0$, we can assume
$B_\varepsilon(x_i) \cap B_\varepsilon(x_j) = \varnothing$ for $i \neq j$,
$B_\varepsilon(x_i) \cap B_\varepsilon(y_j) = \varnothing$ for any i, j,
$B_\varepsilon(y_i) \cap B_\varepsilon(y_j) = \varnothing$ for $i \neq j$,
$\{\exp(rS_xM);\ x \in B_\varepsilon(x_i)\} \cap B_\varepsilon(x_j) = \varnothing$ for $i \neq j$,
$\{\exp(rS_yM);\ y \in B_\varepsilon(y_i)\} \cap B_\varepsilon(y_j) = \varnothing$ for $i \neq j$,
$\{\exp(rS_xM);\ x \in B_\varepsilon(x_i)\} \cap B_\varepsilon(y_j) = \varnothing$ for $i \neq j$,

$\{\exp(rS_yM); y \in B_\varepsilon(y_i)\} \cap B_\varepsilon(x_j) = \varnothing$ for $i \neq j$, so that

$$\langle \varphi_i, \varphi_j\rangle = 0, \quad \langle \varphi_i, \psi_j\rangle = 0, \quad \langle \psi_i, \psi_j\rangle = 0,$$
$$\langle L_r\varphi_i, \varphi_j\rangle = 0, \quad \langle L_r\varphi_i, \psi_j\rangle = 0,$$
$$\langle L_r\psi_i, \varphi_j\rangle = 0, \quad \langle L_r\psi_i, \psi_j\rangle = 0.$$

Since $(L_r\varphi_i)(y_i) > 0$, $(L_r\psi_i)(x_i) > 0$, we have $\langle L_r\varphi_i, \psi_i\rangle > 0$. Putting $\Phi_i = \varphi_i + \psi_i$, we get the desired functions.

Next we deal with negative eigenvalues. Take a set $\{x_i, y_i\}$ in the same way as above. The assumption that there is no geodesic loop with length r means that $\exp(rS_{x_i}M) \not\ni x_i$, $\exp(rS_{y_i}M) \not\ni y_i$. Hence we can assume, in addition,

$$\{\exp(rS_xM); x \in B_\varepsilon(x_i)\} \cap B_\varepsilon(x_i) = \varnothing, \ \{\exp(rS_yM); y \in B_\varepsilon(y_i)\} \cap B_\varepsilon(y_i) = \varnothing,$$

so that

$$\langle L_r\varphi_i, \varphi_i\rangle = 0, \quad \langle L_r\psi_i, \psi_i\rangle = 0.$$

Putting $\Psi_i = \varphi_i - \psi_i$, we obtain

$$\langle \Psi_i, \Psi_j\rangle = 0, \quad \langle L_r\Psi_i, \Psi_j\rangle = 0 \quad \text{for } i \neq j,$$
$$\langle L_r\Psi_i, \Psi_i\rangle < 0 \quad \text{for any } i,$$

which guarantees the existence of infinitely many eigenvalues.

REMARK. We can apply the argument in §2 and this section to the operator M_r (see the remark after Lemma 2.1). The same statements as Theorems A(ii) and D(ii) hold for M_r. (Note that M_r is well defined if (and only if) $\pi_r: SM \to M \times M$ is an immersion.)

5. Expansion of L_r as $r\downarrow 0$. In this section we consider a formal Taylor expansion of L_r with respect to the radius r. The method we will take is different from **[7]**.

We denote by Z the vector field on SM generating the geodesic flow $\{\varphi_t\}$. For any C^∞-function f on M, the function $r \mapsto L_rf(x)$ is smooth, and

$$\frac{d^k}{dr^k}\Big|_{r=0} L_rf(x) = \int_{S_xM} (Z^k\pi^*f)(v)\, dS_x(v).$$

In order to compute more explicitly the function $Z^k(\pi^*f)$, let us choose a local coordinate $(x^1, \ldots, x^n)$ of M, and by Γ^i_{jk} we denote the Christoffel symbols. We write ∇_α for $\nabla_{\partial/\partial x^\alpha}$.

LEMMA 5.1. *Let $v = \sum v^i(\partial/\partial x^i)$. Then*

$$Z^k(\pi^*f)(v) = \nabla^k_{v,\ldots,v}f = \sum_{\alpha_1\cdots\alpha_k} v^{\alpha_1}\cdots v^{\alpha_k}\nabla_{\alpha_k}\cdots\nabla_{\alpha_1}f.$$

PROOF. For a local coordinate of TM, choose $(x^1, \ldots, x^n, v^1, \ldots, v^n)$. The vector field Z is then expressed as

$$\sum v^i\frac{\partial}{\partial x^i} - \sum \Gamma^i_{jk}v^jv^k\frac{\partial}{\partial v^i}$$

(see S. Sasaki **[18]**), so we find

$$Z(\pi^*f)(v) = \sum v^i\nabla_i f.$$

Supposing now that we have proved the assertion for k, we obtain

$$Z(Z^k\pi^*f)(v) = \sum v^i v^{\alpha_1} \dots v^{\alpha_k} \frac{\partial}{\partial x^i} \nabla_{\alpha_k} \cdots \nabla_{\alpha_1} f$$

$$- \sum_{i=1}^{k} \sum \Gamma^{\alpha_i}_{jk} v^j v^k v^{\alpha_1} \cdots \widehat{v^{\alpha_i}} \cdots v^{\alpha_k} \nabla_{\alpha_k} \cdots \nabla_{\alpha_1} f$$

$$= \sum v^{\alpha_1} \cdots v^{\alpha_k} v^i \Bigg(\frac{\partial}{\partial x^i} \nabla_{\alpha_k} \cdots \nabla_{\alpha_1} f - \sum_{h=1}^{k} \Gamma^h_{i\alpha_i} \nabla_{\alpha_k} \cdots \nabla_{\alpha_{i+1}} \nabla_h \nabla_{\alpha_{i-1}} \cdots \nabla_{\alpha_1} f \Bigg)$$

$$= \sum v^{\alpha_1} \cdots v^{\alpha_{k+1}} \nabla_{\alpha_{k+1}} \cdots \nabla_{\alpha_1} f,$$

which implies that the lemma holds for $k + 1$. Hence the proof is completed.

We put

$$h^{\alpha_1 \cdots \alpha_k}(x) = \int_{S_x M} v^{\alpha_1} \cdots v^{\alpha_k} \, dS_x(v),$$

$$P_k = \sum h^{\alpha_1 \cdots \alpha_k} \nabla_{\alpha_k} \cdots \nabla_{\alpha_1}.$$

Since $h^{\alpha_1 \cdots \alpha_k} = 0$ for odd k, the formal Taylor expansion of L_r is given by

$$L_r \sim \sum_{k=0}^{\infty} \frac{1}{(2k)!} P_{2k} r^{2k}.$$

Correctly speaking, we have

LEMMA 5.2. *Let $f \in C^\infty(M)$. For each k, we can write*

$$L_r f = f + \frac{1}{2!} r^2 P_2 f + \cdots + \frac{1}{(2k)!} r^{2k} P_{2k} f + o(r^{2(k+1)}).$$

Here $o(r^{2(k+1)})$ is a C^∞-function on M satisfying

$$\lim_{r \to 0} \frac{1}{r^{2k}} o(r^{2(k+1)}) = 0$$

uniformly on any compact set in M.

REMARK. If (M, g) is real analytic, and if $f \in C^\omega(M)$, then $L_r f(x)$ is a real analytic function of the variable r, and the above expansion converges for small r.

We next calculate the leading symbols of the differential operators P_{2k}. To facilitate doing this, we introduce the following notation. For a formal power series with linear differential operator coefficients, $D_r = \sum_{k=0}^{\infty} r^k D_k$, we define a symbol series $\sigma(D_r)$ to be

$$\sigma(D_r) = \sum_{k=0}^{\infty} \sigma(D_k) r^k,$$

where $\sigma(D_k)$ stands for the leading symbol of D_k.

LEMMA 5.3. *Let* $Q_\nu(x) = 2^\nu \Gamma(\nu + 1) x^{-\nu} J_\nu(x)$. *Then*

$$\sigma\left(\sum_{k=0}^{\infty} \frac{1}{(2k)!} P_{2k} r^{2k}\right) = \sigma\left(Q_{(n-2)/2}(\sqrt{-\Delta}\, r)\right).$$

Here the right-hand side stands for the symbol series of what is obtained by expanding $Q_\nu(\sqrt{-\Delta}\, r)$ $(\nu = (n-2)/2)$, *into a series of powers of* Δ.

PROOF. Let $\xi = \sum \xi_i\, dx^i$ be any covector in $T_x^* M$. Since

$$(P_{2k})(\xi) = \sum h^{\alpha_1 \cdots \alpha_{2k}} \xi_{\alpha_1} \cdots \xi_{\alpha_{2k}},$$

the left side equals

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} r^{2k} \sum \int_{S_x M} v^{\alpha_1} \xi_{\alpha_1} \cdots v^{\alpha_{2k}} \xi_{\alpha_{2k}}\, dS_x(v)$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!} r^{2k} \int_{S_x M} (v^1 \xi_1 + \cdots + v^n \xi_n)^{2k}\, dS_x(v).$$

By a well-known formula (cf. F. John **[12]**), the integral becomes

$$\frac{\Gamma(n/2)\Gamma\left(k + \frac{1}{2}\right)}{\sqrt{\pi}\, \Gamma((n+2k)/2)} \|\xi\|^{2k},$$

so that the last expression is equal to

$$\Gamma(n/2)\left(\sqrt{-1}\, \|\xi\| r/2\right)^{-(n-2)/2}$$

$$\cdot \sum_{k=0} \frac{(-1)^k}{k!\, \Gamma((n-2)/2 + k + 1)} \left(\frac{\sqrt{-1}\, \|\xi\| r}{2}\right)^{(n-2)/2 + 2k}$$

$$= Q_{(n-2)/2}\left(\sqrt{-1}\, \|\xi\| r\right).$$

Since the leading symbol of the Laplacian Δ is $-\|\xi\|^2$, we have proved the lemma.

COROLLARY. P_{2k} *is a formally selfadjoint elliptic differential operator whose principal symbol coincides with that of*

$$\frac{\Gamma(n/2)(2k)!}{k!\, \Gamma((n-2)/2 + k + 1) 2^{2k}} (-\Delta)^k.$$

A direct computation yields

$$P_2 f = -(1/n)\Delta f,$$

$$P_4 f = \frac{3}{n(n+2)}\left(\Delta^2 f + \tfrac{1}{3}\langle \nabla f, \nabla \tau\rangle + \tfrac{2}{3}\langle \nabla^2 f, \rho\rangle\right),$$

where ρ is the Ricci curvature, and τ denotes the scalar curvature (cf. **[7]**).

We now suppose M is compact, and let

$$K_t f(x) = \int_M k_t(x, y) f(y)\, dg(y),$$

$k_t(x, y)$ being the fundamental solution of the heat operator $\partial/\partial t + \Delta$. From the probabilistic view point, K_t is considered as the transition operator associated with the Brownian motions on M.

To prove Theorem E, we recall a theorem of P. Chernoff [4], which says that if a one-parameter family $F(t)$ $(t \geqslant 0)$ of bounded linear operators of a Banach space $\mathfrak{X}$ satisfies $\|F(t)\| \leqslant 1$, $0 \leqslant t < \infty$, $F(0) = I$, and there exists an infinitesimal generator A of a contraction semigroup R^t on $\mathfrak{X}$ such that

$$\lim_{t\to 0} \frac{F(h) - I}{h} u = Au$$

for each u in a core of A, then for all u in $\mathfrak{X}$,

$$\lim_{N\to\infty} F(t/N)^N u = R^t u.$$

In view of Lemmas 2.1 and 5.2, this theorem can be applied to the cases $\mathfrak{X} = L^2(M)$ or $C^0(M)$, $F(t) = L_{\sqrt{2nt}}$, $A = \Delta$, $R^t = K_t$, thereby proving Theorem E.

REMARK. C^0-convergence comes from a more general result proved by Jorgensen [13].

The following two lemmas are obvious from the definition of L_r.

LEMMA 5.4. *Let φ be an isometry of M. Then for any $r > 0$, $L_r\varphi^* = \varphi^* L_r$.*

LEMMA 5.5. *Let $\tilde{\omega}$: $\tilde{M} \to M$ be a Riemannian covering. Let $\tilde{L}_r$ be the spherical mean on $\tilde{M}$. Then $\tilde{L}_r\tilde{\omega}^* = \tilde{\omega}^* L_r$.*

PROPOSITION 5.6. *Let M be a complete locally Riemannian symmetric space. Then $L_r\Delta = \Delta L_r$.*

PROOF. In view of Lemma 5.5, we may assume, without loss of generality, that M is globally symmetric, so that we may write $M = G/H$, where G is the largest connected group of isometries acting on M. From commutativity of L_r with each element in G, it follows that P_{2k} is a G-invariant differential operator. We know from general theory of symmetric spaces [10] that G-invariant differential operators on M commute with each other, hence $L_r\Delta$ and ΔL_r have the same expansion with respect to r. We note here that M is a C^ω-manifold, so that for any $f \in C^\omega(M)$ we have $L_r\Delta f = \Delta L_r f$. Since $C^\omega(M)$ is dense in $C^\infty(M)$, we get the lemma.

REMARK. The above proof implies $L_rL_s = L_sL_r$ for any s, $r > 0$.

We can now prove Theorem F. Let M be a symmetric space with rank $\geqslant 2$, and let $f = \Sigma f_\lambda$ be the L^2-expansion of $f \in L^2(M)$ into eigenfunctions of the Laplacian. If $L_r f = f$, it follows from the above lemma that $L_r f_\lambda = f_\lambda$ for any λ. Since $f_\lambda \in C^\infty$, it suffices to show that any two points can be joined by an r-geodesic chain. But this is clear from the fact that two points lie always in a two-dimensional totally geodesic flat subspace in M, and any two points in a two-dimensional flat space can be joined by an r-geodesic chain, irrespective of the value r.

Let us consider the case M is a rank one symmetric space of compact type. Denote by l the common length of primitive closed geodesics. Let $v \in S_xM$, and let c be a closed geodesic with $\dot{c}(0) = v$. Choosing an orthogonal basis $v_1, \ldots, v_n$ of T_xM such that $v = v_1$ and v_i is the eigenvector of the linear transformation of T_xM given by $u \to R(v, u)v$, we extend these to the orthogonal frame $v_i(t)$ along c

by the parallel translation. Then any Jacobi field X with $X(0) = 0$ can be written in the form

$$X(t) = \sum_{i=2}^{n} a_i \sin\left(\sqrt{\lambda_i}\, t\right) v_i(t),$$

where $R(v, v_i)v = \lambda_i v_i$. We can assume the following relations for each case (after normalizing the curvature).

M	l	λ_i
S^n $(n \geqslant 2)$	2π	$\lambda_1 = \cdots = \lambda_{n-1} = 1$
$P^n(\mathbf{R})$ $(n \geqslant 2)$	π	:
$P^n(\mathbf{C})$	2π	$\lambda_1 = 1, \lambda_2 = \cdots = \lambda_{2n-1} = \frac{1}{4}$
$P^n(\mathbf{Q})$	2π	$\lambda_1 = \lambda_2 = \lambda_3 = 1, \lambda_4 = \cdots = \lambda_{4n-1} = \frac{1}{4}$
$P^2(\mathbf{C}_a)$	2π	$\lambda_1 = \cdots = \lambda_7 = 1, \lambda_8 = \cdots = \lambda_{15} = \frac{1}{4}$

Let $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{Q}$ or $\mathbf{C}_a$. Any two totally geodesic $\mathrm{codim}_{\mathbf{K}}$-one submanifolds in the projective space $P^n(\mathbf{K})$ have nonempty intersection. This implies that any two points in $P^n(\mathbf{K})$ can be joined by an $(l/2)$-geodesic chain. Hence, in conjunction with the above table we have

PROPOSITION 5.7. *The spherical mean L_r on $P^n(\mathbf{K})$ (resp. on S^n) is ergodic if and only if $r \notin \mathbf{Z}l$ (resp. $r \notin \mathbf{Z}\frac{1}{2}l$).*

REMARK.

$$L_r = L_{r-kl/2} \quad \text{if } k = [2r/l] \text{ is even,}$$
$$L_r = L_{(k+1)/2-r} \quad \text{if } k = [2r/l] \text{ is odd,}$$
$$L_{kl} = \mathrm{Id}.$$

We conclude this section by proposing the following open problem.

PROBLEM. Let E denote the set of positive numbers r such that L_r is ergodic. Then is E always dense in $\mathbf{R}_+$?

6. Generalizations and remarks. This section gives generalizations of some results concerning r-geodesic chains. Since it seems unnecessary to repeat all the argument here, only frameworks will be stated.

Let M be a compact Riemannian manifold, and let $\{\mu_x\}$ $(x \in M)$ be a family of probability measure such that each μ_x is assigned to the tangent space T_xM, which is supposed to satisfy:

(†) $\{\mu_x\}$ is invariant under parallel translations,

that is, for any piecewise smooth curve γ: $[a, b] \to M$, the parallel translation P_γ: $(T_{\gamma(a)}M, \mu_{\gamma(a)}) \to (T_{\gamma(b)}M, \mu_{\gamma(b)})$ along γ is measure preserving.

This condition is equivalent to: for any geodesic curve γ the P_γ is measure preserving. In fact any piecewise smooth curve can be approximated by broken geodesics.

We let N_x, $x \in M$, be supports of measures μ_x such that $P(N_{\gamma(a)}) = N_{\gamma(b)}$ for any curve γ: $[a, b] \to M$. We denote by $\mathcal{C}(\mu)$ the totality of geodesic chains c: $[0, \infty) \to M$ with $\dot{c}(k) = dc(k)/dt \in N_{c(k)}$. In the same vein as §3, we can identify $\mathcal{C}(\mu)$ with the fiber space N whose fiber is $\mathbf{N}_x = N_x \times N_x \times \dots$, and can equip a measure P_μ on $\mathcal{C}(\mu)$. The condition (†) guarantees an analogous result to Lemma 3.1, from which it follows at once that the stochastic process π_k: $\mathcal{C}(\mu) \to M$ given by $\pi_k(c) = c(k)$ has a Markov property

$$P_\mu(T^{-k}\mathbf{A}|\mathfrak{B}_k) = P_\mu(\mathbf{A}|\pi_k) \quad \text{for } \mathbf{A} \in \mathfrak{B},$$

where T: $\mathcal{C}(\mu) \to \mathcal{C}(\mu)$ is the shift, $\mathfrak{B}_k$ is the σ-ring generated by $\pi_k^{-1}(\cdot)$, $\mathfrak{B} = \bigcup \mathfrak{B}_k$, and the symbol $P_\mu(\,|\,)$ stands for the conditional probability.

Letting

$$(L_\mu f)(x) = \int_{T_xM} f(\exp v)\, d\mu_x(v) = \int_{N_x} f(\exp v)\, d\mu_x(v),$$

we have

$$P_\mu(T^{-1}\mathbf{A}|x) = P_{\mu,x}(T^{-1}\mathbf{A}) = \big(L_\mu P_\mu;\, (\mathbf{A})\big)(x),$$

so that $P_{\mu,x}(\pi_k^{-1}A) = (L_\mu^k \chi_A)(x)$ for each k. Usually $P_{\mu,x}(\pi_k^{-1}A)$ is called the k-step transition probability of the Markov process $\{\pi_k\}$ and frequently written as $P_k(x; A)$. Using the same argument as §3, we can prove that if the fiber product measure μ on N is invariant under the geodesic flows, then the shift T is measure preserving, and that the dynamical system $(\mathcal{C}(\mu), P, T)$ is ergodic if and only if 1 is a simple eigenvalue of L_μ.

Example (i). For each positive r, we have a measure μ_x given by

$$\mu_x(A) = \int_{S_xM \cap (A/r)} dS_x.$$

It is easy to see that $\mathcal{C}(\mu)$ is identified with $\mathcal{C}_\infty(r; M)$, and $L_\mu = L_r$.

Example (ii). Define a measure by

$$\mu_x(A) = \mathrm{Vol}(B_x^r)^{-1} \int_{A \cap B_x^r} dv,$$

where $B_x^r = \{v \in T_xM;\ \|v\| \leqslant r\}$. The family $\{\mu_x\}$ obviously satisfies the condition (†). In this case, N is the ball bundle $B^r = \bigcup B_x^r$ on M, and the measure μ is invariant under the geodesic flows.

Let it be assumed that radius r is smaller than the injective radius of M, so that the mapping $\Pi(v) = (\pi v, \exp v)$ is a diffeomorphism of B^r into $M \times M$. Since the distribution kernel K_μ of the associated operator L_μ is given by

$$\langle K, h\rangle = \int_{B^r} \Pi^* h\, dB^r \qquad \big(dB^r = \mathrm{Vol}(B_x^r)^{-1}\, dv\, dg\big),$$

K_μ is an L^2-function on $M \times M$, and hence it follows that L_μ is of Hilbert-Schmidt type. Further we easily contend that $L_\mu(L^2(M)) \subset C^0(M)$. Thus L_μ is ergodic.

EXAMPLE (iii). Put

$$\mu_x(A) = (4\pi r)^{-n/2} \int_A \exp(-\|v\|^2/4r)\, dv,$$

which satisfies (†), and μ is invariant under the geodesic flow. When M has no conjugate point, the associated operator L_μ has the smooth distribution kernel, because the mapping Π: $TM \to M \times M$ ($v \mapsto (\pi v, \exp v)$) is a covering mapping. From this fact, we can prove that L_μ is ergodic.

EXAMPLE (iv). Another important example arises in the theory of symmetric spaces. Let $M = G/K$ be a globally symmetric space. Fix an element a in G. We let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ corresponding to G, and write $a = \exp(rv_0)$, $v_0 \in \mathfrak{g}$ with $\|v_0\| = 1$. Putting $N_{gK} = dg(\mathrm{Ad}(K)v_0)$, we have a subbundle $N = \bigcup N_{gK}$ of SM. We set, for $A \subset \mathfrak{p} = T_{eK}M$,

$$\mu_{eK}(A) = \int_{\{k \in K;\ \mathrm{Ad}(k)v_0 \in A\}} dk,$$

where dk denotes the normalized Haar measure on K. The measure can be extended in an obvious way to a measure on each fiber of N, and the family μ_x satisfies the condition (†), thanks to the symmetricity. Further we easily find

$$(L_\mu f) = \int_K f(gkaK)\, dk.$$

If M is a rank one symmetric space, K acts transitively on the sphere $\{v \in \mathfrak{p};\ \|v\| = 1\}$, so one concludes that L_μ coincides with the spherical mean L_r.

REMARK. As was pointed out in the introduction, there is another way to construct a Markov process with the transition probability $P_k(x, A) = (L_\mu^k \chi_A)(x)$, which is standard to probabilist, but not geometrical. Namely, for a probability space, we choose an infinite product $M^\infty = M \times M \times \ldots$, and define a probability measure P on M^∞ so that, for any $k > 2$, and for any Borel set $A_0, A_1, \ldots, A_k$ of M, the following relation holds:

$$P(X_0 \in A_0, X_1 \in A_1, \ldots, X_k \in A_k) = \int_{A_0} dg(x_0) \int_{A_1} P_1(x_0, dx_1) \ldots \int_{A_k} P_1(x_{k-1}, dx_k)$$

where x_i: $M \to M$ is the ith coordinate. It is a standard fact that the coordinate process $\{x_k\}$ is a Markov process with transition operator L_μ (see [5]). Usually, we call such a process "random walks" because the process provides us with a mathematical model for motions of drunken walkers who walk very randomly on the manifold. The process $(\mathcal{C}(\mu), \pi_k)$ constructed above is considered as a geometrical model of random walks.

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Mathematisches Institut, Universität Bonn, 5300 Bonn, West Germany

Current address: Department of Mathematics, Nagoya University, Nagoya 464, Japan